Comment on Ricci Collineations for spherically symmetric space-times

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Abstract

It is shown that the results of the paper by Contreras et al. [Contreras, G., Nunez, L. A., Percoco, U. *Ricci Collineations for Non-degenerate, Diagonal and Spherically Symmetric Ricci Tensors* (2000). *Gen. Rel. Grav.* **32**, 285-294] concerning the Ricci Collineations in spherically symmetric space-times with non-degenerate and diagonal Ricci tensor do not cover all possible cases. Furthermore the complete algebra of Ricci Collineations of certain Robertson-Walker metrics of vanishing spatial curvature are given.

The determination of Ricci collineations in a spherically symmetric space-time is an interesting and difficult problem. The static case has been considered in a series of papers [1], [2], [3] however in an incomplete way [4]. The non-static case, which is far more difficult and contains the static case as a special case, has been considered recently by Contreras et al. [5]. The purpose of the present comment is to show by counter examples that this later approach although correct it is also incomplete.

In [5] the authors consider the Spherically Symmetric (SS) metric which in natural coordinates has the form:

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

and they look for Ricci Collineations (RCs) X^a defined by $\mathcal{L}_{\mathbf{X}} R_{ab} = 0$ where R_{ab} is the Ricci tensor associated with the metric (1). To do this they use the following result of [6]:

The proper RCs of the space-time (1) whose Ricci tensor is non-degenerate, are of the form:

$$\boldsymbol{\xi} = \xi^t(t,r)\partial_t + \xi^r(t,r)\partial_r \tag{2}$$

and they conclude that "the form of the most general RC vector is the one given in (2) plus linear combinations, with constant coefficients, of the Killing vectors for spherical symmetry". Subsequently they compute 64 different classes of SS metrics which admit (not necessarily all proper) RCs and classify them in Table 1 using the time and the radial first derivatives of the components of the Ricci tensor.

However it has been shown [7] that the form (2) of the generic RC in SS metrics is not the most general one. Consequently one should expect that the results of [5] are also incomplete. This is indeed the case as we show by the following two counter examples based on the very examples given in [5].

Consider the RW metric with $k = 0$:

$$ds^2 = -dt^2 + F^2(t)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{3}$$

The components of the Ricci tensor are:

$$R_{00} = -3\frac{F_{,tt}}{F}, \quad R_{11} = \Delta, \quad R_{22} = r^2\Delta, \quad R_{33} = r^2\Delta\sin^2\theta \tag{4}$$

where:

$$\Delta = FF_{,tt} + 2\left(F_{,t}\right)^2 \tag{5}$$

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and a comma denotes partial derivative w.r.t the index that follows. According to the classification given in [5] the Ricci tensor (4) belongs to the family number 7. One distinguishes two cases according to the constancy of Δ.

Case $\Delta = const. = \pm a^2 \neq 0$

In [5] for this case one finds the RC $\mathbf{X}_1 = e^{\epsilon a^2} |R_{00}(\tau)|^{-1/2} \partial_t$ (cf eq. (33)) which has been found previously by Green et al. [8]. However it is easy to show that the following vector fields are *proper* RCs:

$$\mathbf{X}_2 = ar\cos\phi\sin\theta\partial_{\tilde{\tau}} - \epsilon a^{-1}\tilde{\tau}(t)\left\{\cos\phi\left[\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\right] - \frac{\sin\phi}{r\sin\theta}\partial_\phi\right\} \tag{6}$$

$$\mathbf{X}_3 = ar\sin\phi\sin\theta\partial_{\tilde{\tau}} - \epsilon a^{-1}\tilde{\tau}(t)\left\{\sin\phi\left[\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\right] + \frac{\cos\phi}{r\sin\theta}\partial_\phi\right\} \tag{7}$$

$$\mathbf{X}_4 = ar\cos\theta\partial_{\tilde{\tau}} - \epsilon a^{-1}\tilde{\tau}(t)\left[\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right] \tag{8}$$

where:

$$\tilde{\tau}(t) = \int |R_{00}|^{1/2}\, dt \tag{9}$$

and $\epsilon = sign(R_{00}\Delta)$.

It is well known that if the Ricci tensor is everywhere of rank 4 (i.e. non degenerate) the family of (at least C^2) RCs is a Lie algebra of smooth vector fields of maximum dimension 10 [9]. Hence the 6 KVs of the RW space-time plus the 4 RCs $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ above generate the *complete* Lie algebra of RCs of this case.

Case $\Delta_{,t} \neq 0$

The metric (3) with $F(t) = (M/2b)u^2$ and $u = (6t/M)^{1/3}$ where M and b are constants [10] is a second example in [5]. In this case $\Delta_{,t} \neq 0$. According to the classification given in [5] this space-time admit the RC:

$$\boldsymbol{\xi} = \sqrt{\frac{3}{2}}t\partial_t + \frac{r}{\sqrt{6}}\partial_r. \tag{10}$$

which, as it is correctly stated, is not a proper RC but a HVF . Again it can be shown that the following *proper* RCs complete the Lie algebra of RCs for this space-time:

$$\mathbf{X}_1 = 6rt\cos\phi\sin\theta\partial_t + \frac{\left[r^2M^{2/3} - (6t)^{2/3}b^2\right]\cos\phi\sin\theta}{M^{2/3}}\partial_r - \frac{\left[r^2M^{2/3} + (6t)^{2/3}b^2\right]}{M^{2/3}r}\left[\cos\phi\cos\theta\partial_\theta - \frac{\sin\phi}{\sin\theta}\partial_\phi\right] \tag{11}$$

$$\mathbf{X}_2 = 6rt\sin\phi\sin\theta\partial_t + \frac{\left[r^2M^{2/3} - (6t)^{2/3}b^2\right]\sin\phi\sin\theta}{M^{2/3}}\partial_r - \frac{\left[r^2M^{2/3} + (6t)^{2/3}b^2\right]}{M^{2/3}r}\left[\sin\phi\cos\theta\partial_\theta + \frac{\cos\phi}{\sin\theta}\partial_\phi\right] \tag{12}$$

$$\mathbf{X}_3 = 6rt\cos\theta\partial_t + \frac{\left[r^2M^{2/3} - (6t)^{2/3}b^2\right]\cos\theta}{M^{2/3}}\partial_r + \frac{\left[r^2M^{2/3} + (6t)^{2/3}b^2\right]\sin\theta}{M^{2/3}r}\partial_\theta. \tag{13}$$

From the above we draw the following conclusions:

1. The generic form of the RCs of the SS metric (1) for the case where the Ricci tensor is *non degenerate and diagonal* is:

$$\mathbf{X} = X^t(t,r,\theta,\phi)\partial_t + X^r(t,r,\theta,\phi)\partial_r + X^\theta(t,r,\theta,\phi)\partial_\theta + X^\phi(t,r,\theta,\phi)\partial_\phi. \tag{14}$$

2. The results in [5] do not cover all possible cases and the question of determining all RCs of a SS metric (1) is still open. However this conclusion in no way diminishes the essential contribution of [5].

A complete study of the RCs in RW space-times is under preparation.

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